First Iranian Geometry Olympiad

September 2014

Solutions of Junior Level

1. In a right triangle ABC we have $\angle A = 90^\circ$,$\angle C = 30^\circ$. Denot by C the circle passing through A which is tangent to BC at the midpoint.Assume that C intersects AC and the circumcircle of ABC at N and M respectively. Prove that $MN \perp BC$.

Mahdi Etesami Fard

Proof. Let K midpoint of side BC. Therefore:

$$AK = KC \Rightarrow \angle KAC = \angle NKC = 30^\circ$$

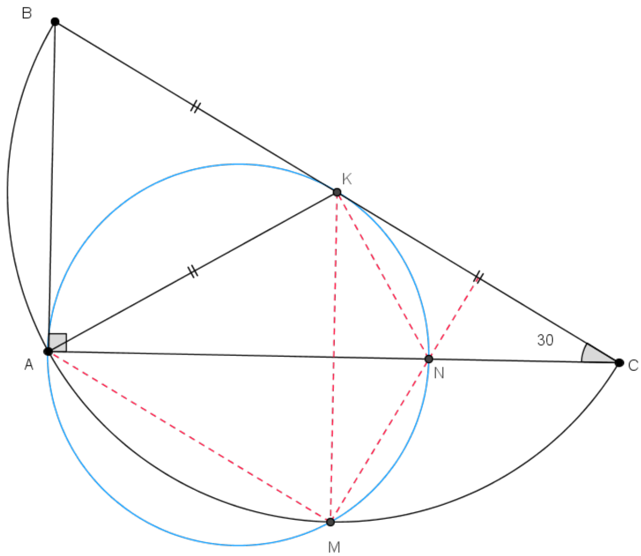
$$\angle ANK = \angle NKC + \angle ACB = 60^\circ$$

A, K, N, M lie on circle (C). Therefore:

$$\angle KAN = \angle KMN = 30^\circ, \angle AMK = 60^\circ$$

We know that K is the circumcenter of $\triangle ABC$. So we can say $KM = KC = AK$. Therefore $\triangle AKM$ is equilateral.(because of $\angle AMK = 60^\circ$). So $\angle AKM = 60^\circ$. We know that $\angle AKB = 60^\circ$, so we have $\angle MKC = 60^\circ$. On the other hand:

$$\angle KMN = 30^\circ \Rightarrow MN \perp BC$$


B
K
30
A
N
C
M

□

2. The inscribed circle of $\triangle ABC$ touches BC, AC and AB at D, E and F respectively. Denote the perpendicular foots from F, E to BC by K, L respectively. Let the second intersection of these perpendiculars with the incircle be M, N respectively. Show that $\frac{S_{\triangle BMD}}{S_{\triangle CND}} = \frac{DK}{DL}$.

Mahdi Etesami Fard

Proof. Let I be the incenter of $\triangle ABC$. We know that

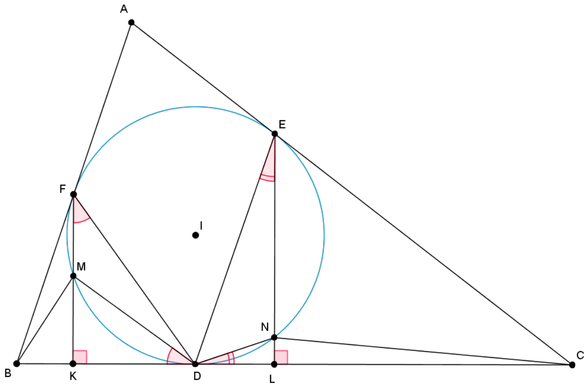
$$\left.\begin{array}{l} \angle BFK = 90^\circ - \angle B \\ \angle BFD = 90^\circ - \frac{1}{2}\angle B \end{array}\right\} \Rightarrow \angle DFM = \frac{1}{2}\angle B$$

But $\angle DFM = \angle MDK$. Therefore

$$\angle MDK = \frac{1}{2}\angle B$$

hence $\triangle MDK$ and $\triangle BID$ are similar (same angles) and $\frac{MK}{DK} = \frac{r}{BD}$. In the same way we have $\frac{NL}{DL} = \frac{r}{CD}$. Therefore

$$r = \frac{MK \cdot BD}{DK} = \frac{NL \cdot CD}{DL} \Rightarrow \frac{\text{area of } \triangle BMD}{\text{area of } \triangle CND} = \frac{MK \cdot BD}{NL \cdot CD} = \frac{DK}{DL}$$


A
E
F
I
M
N
B
K
D
L
C

□

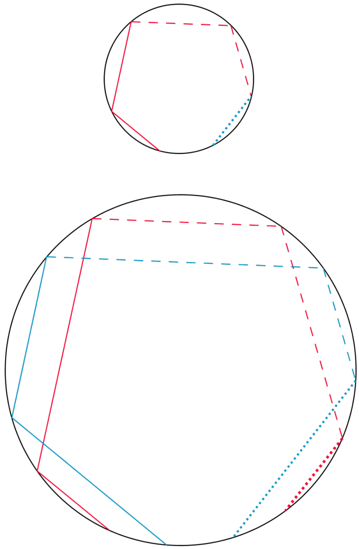
3. Each of Mahdi and Morteza has drawn an inscribed 93-gon. Denote the first one by $A_1A_2...A_{93}$ and the second by $B_1B_2...B_{93}$. It is known that $A_iA_{i+1} \parallel B_iB_{i+1}$ for $1 \leqslant i \leqslant 93$ ($A_{93} = A_1, B_{93} = B_1$). Show that $\frac{A_iA_{i+1}}{B_iB_{i+1}}$ is a constant number independent of i.

Morteza Saghafian

Proof. We draw a 93-gon similar with the second 93-gon in the circumcircle of the first 93-gon (so the sides of the second 93-gon would be multiplying by a constant number c). Now we have two 93-gons witch are inscribed in the same circle and apply the problem's conditions. We name this 93-gons $A_1A_2...A_{93}$ and $C_1C_2...C_{93}$.

We know that $A_1A_2 \parallel C_1C_2$. Therefore $\overset{\frown}{A_1C_1}=\overset{\frown}{A_2C_2}$ but they lie on the opposite side of each other. In fact, $\overset{\frown}{A_iC_i}=\overset{\frown}{A_{i+1}C_{i+1}}$ and they lie on the opposite side of each other for all $1 \leqslant i \leqslant 93$ ($\overset{\frown}{A_{94}C_{94}}=\overset{\frown}{A_1C_1}$). Therefore $\overset{\frown}{A_1C_1}$ and $\overset{\frown}{A_1C_1}$ lie on the opposite side of each other. So $\overset{\frown}{A_1C_1}= 0°$ or $180°$. This means that the 93-gons are coincident or reflections of each other across the center. So $A_iA_{i+1} = C_iC_{i+1}$ for

$1 \leqslant i \leqslant 93$. Therefore, $\frac{A_iA_{i+1}}{B_iB_{i+1}} = c$.



$\square$

4. In a triangle ABC we have $\angle C = \angle A + 90^\circ$. The point D on the continuation of BC is given such that $AC = AD$. A point E in the side of BC in which A doesnt lie is chosen such that

$$\angle EBC = \angle A, \angle EDC = \frac{1}{2}\angle A$$

Prove that $\angle CED = \angle ABC$.

Morteza Saghafian

Proof. Suppose M is the midpoint of CD. hence AM is the perpendicular bisector of CD. AM intersects DE and BE at P, Q respectively. Therefore, $PC = PD$. We have

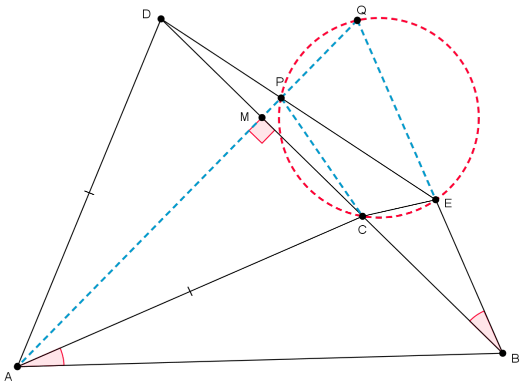
$$\angle EBA + \angle CAB = \angle A + \angle B + \angle A = 180^\circ - \angle C + \angle A = 90^\circ$$

hence $AC \perp BE$. Thus in $\triangle ABQ$, BC, AC are altitudes. This means C is the orthocenter of this triangle and

$$\angle CQE = \angle CQB = \angle A = \frac{1}{2}\angle A + \frac{1}{2}\angle A = \angle PDC + \angle PCD = \angle CPE$$

hence $CPQE$ is cyclic. Therefore

$$\angle CED = \angle CEP = \angle CQP = \angle CQA = \angle CBA = \angle B.$$



D
Q
P
M
E
C
A
B

□

5. Two points X, Y lie on the arc BC of the circumcircle of $\triangle ABC$ (this arc does not contain A) such that $\angle BAX = \angle CAY$. Let M denotes the midpoint of the chord AX . Show that $BM + CM > AY$.

Mahan Tajrobekar

Proof. O is the circumcenter of $\triangle ABC$, so $OM \perp AX$. We draw a perpendicular line from B to OM. This line intersects with the circumcircle at Z. Since $OM \perp BZ$, OM is the perpendicular bisector of BZ. This means $MZ = MB$. By using triangle inequality we have

$$BM + MC = ZM + MC > CZ$$

But $BZ \parallel AX$, thus

$$\overset{\frown}{AZ} = \overset{\frown}{BX} = \overset{\frown}{CY} \Rightarrow Z\widehat{A}C = Y\widehat{C}A \Rightarrow CZ = AY$$

hence $BM + CM > AY$.